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A conjecture concerning the Hadamard product of inverses of M-matrices

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Abstract

We conjecture that for an $n \times n$ matrix A which is an inverse of an M-matrix, the Hadamard product $A \circ A$ is also an inverse of an M-matrix. We have checked this conjecture without failure on many many examples. But here we show that for quite a few well known classes of inverses of M-matrices, the conjecture is true. It is known that the more general conjecture, that when A and B are $n \times n$ inverses of M-matrices, then $A \circ B$ is also an inverse of an M-matrix, is false. However, here too we are able to display some classes of inverses of M-matrices which are closed under taking Hadamard products.

1. Introduction

Various properties concerning the *Hadamard product* (the entrywise product, sometimes also called the *Schur product*) of M-matrices and Hadamard products of matrices closely related to M-matrices are known. For example, Fan in [4] and, independently, Lynn in [13] have shown that if C and D are nonsingular M-matrices, then the comparison matrix $\mathscr{M}(C \circ D)$ is again a nonsingular M-matrix. Another property of Hadamard matrices due to Fiedler and Pták [5] (see also Johnson [8]) is that if C and D are nonsingular M-matrices, then $C \circ D^{-1}$ is an M-matrix. A further property of

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interest is due to Ando [1]. He shows that is that if $(I - S)$ and $(I - T)$ are nonsingular M-matrices, then as nonnegative matrices

$$(I - S \circ T)^{-1} \leqslant (I - S)^{-1} \circ (I - T)^{-1},$$

the inequality being entrywise.

Here we conjecture a different property concerning the Hadamard product of matrices closely related to M-matrices.

Conjecture 1.1. If A is an $n \times n$ matrix which is an inverse of an M-matrix, then $A \circ A$ is an inverse of an M-matrix.

It is quite easy to show that if A is an inverse of an M-matrix and $n \leqslant 3$, the conjecture is true. For $n = 3$, for example, the sign of all the 2×2 minors in A and in $A \circ A$ are identical. For $n > 3$ we have checked this conjecture, without failure, on many many examples. But for the time being we shall show that the conjecture is true for several well known classes of inverses of M-matrices. Each class will be examined in section by itself. It is known, via an example due to Johnson, Markham, and Neumann (see Johnson [9]), that the more general conjecture, that if A and B are inverses of M-matrices, then $A \circ B$ is an inverse of an M-matrix, is false. However, we shall exhibit some classes of inverses of M-matrices which are closed under taking Hadamard product.

Before we proceed with the above plan, let us mention that, as illustrated in [24], without loss of generality, we can assume that the nonnegative matrix $A = (a_{i,j})$, whose properties we wish to investigate in order to determine whether it is an inverse of an M-matrix, *has unit diagonal entries and off-diagonal entries which are bounded above by* 1. In many of our statements here we shall tacitly assume that the elements of A satisfy these requirements and sometimes refer to such a matrix A as *normalized.* Furthermore, in his Theorem 1 (see p. 79), Willoughby shows that *necessary* conditions for A to be an inverse of an M-matrix are that for all distinct triplets of indices (i, j, k), $1 \leqslant i, j, k \leqslant n$,

$$a_{i,j} - a_{i,k}a_{k,j} \geqslant 0, \tag{1.1}$$

$$a_{i,i}a_{k,k} - a_{i,k}a_{k,i} = 1 - a_{i,k}a_{k,i} > 0. \tag{1.2}$$

Thus, clearly, if A is an inverse of an M-matrix and $B = (b_{i,j}) = A \circ A$, then for all distinct triplets of indices (i, j, k), $1 \leqslant i, j, k \leqslant n$,

$$b_{i,j} - b_{i,k}b_{k,j} = a_{i,j}^2 - a_{i,k}^2 a_{k,j}^2 \geqslant 0, \tag{1.3}$$

$$b_{i,i}b_{k,k} - b_{i,k}b_{k,j} = 1 - a_{i,k}^2 a_{k,i}^2 > 0. \tag{1.4}$$

We therefore see that B satisfies Willoughby's necessary conditions for being an inverse of an M-matrix.

Actually we claim that we can say more about $B = A \circ A$ when A is an inverse of an M-matrix and when we, *additionally*, assume that so is B. To see this we first need to introduce some common concepts from directed graphs.

Let $\langle n \rangle = \{1, 2, \ldots, n\}$. Then a *digraph* $\Gamma = (V, E)$ consists of vertex set V, conveniently labeled from 1 to n, and set of directed edges $E = \{(i,j) \mid i,j \in V\}$. A *path* from j to k in Γ is a sequence of vertices $j = r_1, r_2, \ldots, r_t = k$, with $(r_i, r_{i+1}) \in E$, for $i = 1, \ldots, t-1$. A path is *simple* if $r_1, r_2, \ldots, r_t$ are distinct. A path $r_1, \ldots, r_t, r_1$ with $t > 1$ is called a *cycle*. It is called a *simple cycle* if the intermediate vertices are distinct. The *digraph of a matrix* $A = (a_{i,j}) \in \mathbb{R}^{n,n}$ denoted by $\Gamma(A) = (V, E)$ has $V = \langle n \rangle$ as its vertex set and $E = \{(i,j) \mid a_{i,j} \neq 0\}$ as its edge set. For a general background concerning nonnegative matrices, M-matrices, and directed graphs see Berman and Plemmons [2].

Suppose now that $A \in \mathbb{R}^{n,n}$ is an inverse of an M-matrix. Then, according to Lewin and Neumann ([12], Cor. 1 and p. 45), $\Gamma(A) = \overline{\Gamma(A)}$, where for a directed graph Γ, $\overline{\Gamma}$ is its *reflexive transitive closure*. Now, as A is nonsingular, it follows from Schneider ([21], Lemma 2.2) that

$$\Gamma(A) \subseteq \overline{\Gamma(A^{-1})} \subseteq \overline{\left(\overline{\Gamma(A)}\right)} = \Gamma(A).$$

Thus, if B is also an inverse of an M-matrix, then as $\Gamma(B) = \Gamma(A)$, then that

$$\overline{\Gamma(B^{-1})} = \overline{\Gamma(A^{-1})}. \tag{1.5}$$

We can refine Eq. (1.5). This is so as because in ([18], Theorem 3.9) it is shown that if equality holds in Eq. (1.1), viz.

$$a_{i,j} = a_{i,k} a_{k,j}, \tag{1.6}$$

then in $\Gamma_k(A^{-1})$, which is the subgraph of $\Gamma(A^{-1})$ induced by $\langle n \rangle \setminus \{k\}$, there is *no* path from i to j, while

$$a_{i,j} > a_{i,k} a_{k,j} \tag{1.7}$$

if there is a directed path from i to j in $\Gamma_k(A)$. But then if $B = A \circ A$, Eq. (1.6) holds if and only if

$$b_{i,j} = b_{i,k} b_{k,j}$$

while Eq. (1.7) holds if and only if

$$b_{i,j} > b_{i,k} b_{k,j}$$

holds. This means that if B^{-1} is an M-matrix too, then

$$\overline{\Gamma_k(B^{-1})} = \overline{\Gamma_k(A^{-1})}, \quad k = 1, \ldots, n. \tag{1.8}$$

Despite these facts, the following example shows that even when B is an inverse of an M-matrix, $\Gamma(A^{-1})$ and $\Gamma(B^{-1})$ *need not* coincide. Let

$$A = \begin{bmatrix} 1 & 0.3781 & 0.3781 & 0.2 \\ 0.3781 & 1 & 0.4286 & 0.3781 \\ 0.3781 & 0.4286 & 1 & 0.3781 \\ 0.2 & 0.3781 & 0.3781 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.25 & -0.3308 & -0.3308 & 0 \\ -0.3308 & 1.4 & -0.35 & -0.3308 \\ -0.3308 & -0.35 & 1.4 & -0.3308 \\ 0 & -0.3308 & -0.3308 & 1.25 \end{bmatrix}^{-1}. \tag{1.9}$$

Then for $B = A \circ A$,

$$B = \begin{bmatrix} 1 & 0.1429 & 0.1429 & 0.04 \\ 0.1429 & 1 & 0.1837 & 0.1429 \\ 0.1429 & 0.1837 & 1 & 0.1429 \\ 0.04 & 0.1429 & 0.1429 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.036 & -0.1243 & -0.1243 & -0.005919 \\ -0.1243 & 1.065 & -0.1601 & -0.1243 \\ -0.1243 & -0.1601 & 1.065 & -0.1243 \\ -0.005919 & -0.1243 & -0.1243 & 1.036 \end{bmatrix}^{-1} \tag{1.10}$$

so that $\Gamma(B^{-1}) \neq \Gamma(A^{-1})$ as $(1,4) \in \Gamma(B^{-1})$, while $(1,4) \notin \Gamma(A^{-1})$. Actually we observe too that although, due to our normalization, $A \geqslant B = A \circ A$, when B^{-1} is an M-matrix then *not necessarily*, $B^{-1} \geqslant A^{-1}$. This is examplified by comparing the $(1,1)$-entries in A^{-1} and B^{-1} above. [2]

A final comment here. We see that if $A \in \mathbb{R}^{n,n}$ is an inverse of an M-matrix, then regardless of whether $A \circ A$ is an inverse of an M-matrix, $(A \circ A) \circ F$ is an M-matrix for any M-matrix F. This is because $(A \circ A) \circ F = A \circ (A \circ F)$. Now $A \circ F$ is an M-matrix by the properties of M-matrices mentioned at the beginning of the paper. Whence, applying the property again, $A \circ (A \circ F)$ is an M-matrix. However, from the counterexample found in [9], in which A and B are inverses of M-matrices, but $A \circ B$ is not, we see that despite the fact that it still holds that $(A \circ B) \circ F$ is an inverse of an M-matrix for any M-matrix F, such a multiplicative property is not strong enough to force $A \circ B$ itself to be an inverse of an M-matrix.

[2] It is well known that if A and B are $n \times n$ matrices which are inverses of M-matrices and $B \leqslant A$, then not necessarily $A^{-1} \leqslant B^{-1}$. In this paper though we are explicitly addressing the case when A and B are inverses of M-matrices and $B = A \circ A$.

2. Ultrametric matrices and inverse of MMA-matrices

In [16], Martinez et al. introduce the following class of nonnegative symmetric matrices.

Definition 2.1. An $n \times n$ matrix $A = (a_{i,j})$ is a called an *ultrametric matrix* if
(i) A is symmetric with nonnegative entries,
(ii) $a_{i,j} \geqslant \min\{a_{i,k}, a_{k,j}\}$, for all $i, j, k \in \langle n \rangle$,
(iii) $a_{i,i} \geqslant \max\{a_{i,k} : k \in \langle n \rangle \setminus i\}$, for all $i \in \langle n \rangle$.

We say that A is a *strictly ultrametric matrix* if the inequality in (iii) is strict for all $i \in \langle n \rangle$. If $n = 1$ and $A > 0$, then A is considered to be strictly ultrametric. We note that ultrametric matrices are called *pre-ultrametric* matrices in [22]. In that paper matrices called ultrametric are required to be nonsingular.

Martinez et al. showed that if A is a strictly ultrametric matrix, then A^{-1} is an M-matrix.

Ultrametric matrices were further studied and generalized in several papers, see e.g. [18] and [20]. In [18] the following generalization was introduced.

Definition 2.2. Let $A = (a_{i,j}) \in \mathbb{R}^{n,n}$ and let $\{i, j, k\}$ be distinct integers from $\langle n \rangle := \{1, 2, \ldots, n\}$. We say that i is a *preferred element* of $\{i, j, k\}$ if
(i) $a_{i,j} = a_{i,k}$.
(ii) $a_{j,i} = a_{k,i}$.
(iii) $\min\{a_{j,k}, a_{k,j}\} \geqslant \min\{a_{j,i}, a_{i,j}\}$.
(iv) $\max\{a_{j,k}, a_{k,j}\} \geqslant \max\{a_{j,i}, a_{i,j}\}$.

Definition 2.3. Let $A = (a_{i,j}) \in \mathbb{R}^{n,n}$. We call A a *generalized ultrametric matrix* if:
(i) A is nonnegative.
(ii) $a_{i,i} \geqslant \max\{a_{j,i}, a_{i,j}\}$, for all $i, j \in \langle n \rangle$.
(iii) $n \leqslant 2$ or $n > 2$ and every subset of $\langle n \rangle$ with three distinct elements has a preferred element.

Now for the result from [18].

Theorem 2.4 (see [18], Theorem 4.4). *Let $A \in \mathbb{R}^{n,n}$ be a generalized ultrametric matrix. Then the following are equivalent:*
(a) *A is nonsingular.*
(b) *A does not contain a row of zeros and no two rows of A are identical.*
(c) *A is nonsingular and A^{-1} is a row and column diagonally dominant M-matrix.*

Based on Theorem 2.4 we have the following Theorem 2.5.

Theorem 2.5. *Let $A = (a_{i,j}) \in \mathbb{R}^{n,n}$ be a nonsingular generalized ultrametric matrix and let $f: R_+ \to R_+$, where R_+ denotes the nonnegative reals, be a strictly increasing function. Then the matrix $F = (f(a_{i,j}))$ is an inverse of an M-matrix. In particular, $A \circ A$ is an inverse of an M-matrix.*

Proof. It follows immediately from the definition of a generalized ultrametric matrix and the assumptions about the function f that F is a generalized ultrametric matrix. Next, by the equivalence of (a) and (b) in Theorem 2.4, A cannot have a zero row, nor two rows which are identical. Therefore, the properties of f now assure us that F cannot have a zero row, nor two rows which are identical. Thus, by the equivalence of (b) and (c) in Theorem 2.4, F must be a nonsingular matrix which is an inverse of an M-matrix. The last part of the theorem follows by choosing $f(x) = x^2$. □

Recall the concept, due to Friedland et al. [6], that an $n \times n$ matrix A is called an *MMA-matrix* if A and all its powers are irreducible M-matrices. Recently, Elsner et al. ([3] Theorem 3.1), have shown that if A is an inverse of an MMA-matrix, then there exits a positive diagonal matrix E such that EAE is a strictly ultrametric matrix. In the next theorem we shall refer to a function f as *multiplicative* if $f(xy) = f(x)f(y)$ for all $x, y \in \mathcal{D}(f)$, the domain of f. Thus a consequence of Theorem 2.5 is the following theorem.

Theorem 2.6. *Let A be an $n \times n$* matrix *which is the inverse of an MMA-matrix. Let f be a strictly increasing, nonnegative, multiplicative function on $\mathbb{R}_+$. Then the matrix $F = (f(a_{i,j}))$ is an inverse of an M-matrix. In particular $A \circ A$ is an inverse of an M-matrix.*

Proof. As mentioned above, according to ([3], Theorem 3.1), there exists a positive diagonal matrix $E = (e_{i,j}\delta_{i,j})$ such that the matrix EAE is a strictly ultrametric matrix. Now, the (i,j)-th entry of $D := EAE$ is given by $d_{i,j} = e_{i,i}a_{i,j}e_{j,j}$ and so, due to the multiplicativity of f, $f(e_{i,i}a_{i,j}e_{j,j}) = f(e_{i,i})f(a_{i,j})f(e_{j,j})$, $i,j = 1, \ldots, n$. This shows that

$$F = \big(f(a_{i,j})\big) = \mathscr{E}^{-1}\big(f(d_{i,j})\big)\mathscr{E}^{-1},$$

where $\mathscr{E}$ is a positive diagonal matrix whose ith diagonal entry is given by $f(e_{i,i})$, $i = 1, \ldots, n$. Now, by Theorem 2.5, the matrix $(f(d_{i,j}))$ is an inverse of an M-matrix. Hence, easily, F is also an inverse of an M-matrix. The final part of the theorem follows now by choosing $f(x) = x^2$. □

As an illustration that the assumption concerning the multiplicativity of f in Theorem 2.6 cannot be removed consider the following inverse of an MMA-matrix $A = (a_{i,j})$

$$A = \begin{bmatrix} 0.166704 & 0.0950331 & 0.210721 & 0.0186995 & 0.0123816 \\ 0.0950331 & 0.0675332 & 0.127732 & 0.0113350 & 0.00750532 \\ 0.210721 & 0.127732 & 0.832528 & 0.0617204 & 0.0408674 \\ 0.0186995 & 0.0113350 & 0.0617204 & 0.681752 & 0.0111376 \\ 0.0123816 & 0.00750532 & 0.0408674 & 0.0111376 & 0.900619 \end{bmatrix}$$

$$= \begin{bmatrix} 31.8991 & -41.7244 & -1.66940 & -0.0298495 & -0.0147127 \\ -41.7244 & 75.4403 & -1.01193 & -0.0180938 & -0.00891834 \\ -1.66940 & -1.01193 & 1.78865 & -0.0985229 & -0.0485615 \\ -0.0298495 & -0.0180938 & -0.0985229 & 1.47706 & -0.0132345 \\ -0.0147127 & -0.00891834 & -0.0485615 & -0.0132345 & 1.11299 \end{bmatrix}^{-1}.$$

Now let $f(x) = e^x$. Then we find that

$$(e^{a_{i,j}}) = \begin{bmatrix} 1.18140 & 1.09970 & 1.23457 & 1.01888 & 1.01246 \\ 1.09970 & 1.06987 & 1.13625 & 1.01140 & 1.00753 \\ 1.23457 & 1.13625 & 2.29912 & 1.06366 & 1.04171 \\ 1.01888 & 1.01140 & 1.06366 & 1.97734 & 1.01120 \\ 1.01246 & 1.00753 & 1.04171 & 1.01120 & 2.46113 \end{bmatrix}$$

$$= \begin{bmatrix} 21.5911 & -21.4656 & -1.30569 & 0.408033 & 0.290440 \\ -21.4656 & 24.7570 & 0.301483 & -1.30674 & -0.895156 \\ -1.30569 & 0.301483 & 0.994916 & -0.0162347 & -0.000729990 \\ 0.408033 & -1.30674 & -0.0162347 & 0.989210 & -0.0324672 \\ 0.290440 & -0.895156 & -0.000729990 & -0.0324672 & 0.666944 \end{bmatrix}^{-1}.$$

We comment that it is interesting to contrast the result of Theorem 2.6 with a result in [3] in which it was proved that if f is a positive nondecreasing function and $A \in \mathbb{R}^{n,n}$ is an MMA-matrix, then the matrix $f(A)$ is an M-matrix.

3. Unipathic matrices

The following definition is taken from [19].

Definition 3.1. A digraph is called *unipathic* if there is at most one simple path from any vertex j to any other vertex k. The matrix A is called *unipathic* if $\Gamma(A)$ is a unipathic digraph.

A unipathic digraph may have loops on its vertices and, unlike a digraph whose underlying undirected graph is a tree, it may have cycles of any length.

However, no two cycles can have a common edge. As explained in [17], every strongly connected unipathic digraph can be constructed from a tree (by adjoining chords and orienting the resulting cycles, and by replacing edges with directed simple paths). Notice that if the digraph of a combinatorially symmetric matrix $A = (a_{i,j})$ (i.e., $a_{i,j} \neq 0$ implies $a_{j,i} \neq 0$) is strongly connected and unipathic, then its underlying undirected graph must be a tree.

In [19] McDonald et al. prove the following result.

Theorem 3.2. *Let Γ be a unipathic graph on n vertices and $A \in \mathbb{R}^{n,n}$. Then the following are equivalent:*

(i) *A is nonsingular and A^{-1} is an M-matrix such that $\Gamma(A^{-1}) = \Gamma$.*

(ii) *$A \geqslant 0$ and satisfies:*

(a) *$a_{i,i} > 0$, for all $i \in \langle n \rangle$.*

(b) *$a_{j,j}a_{k,k} > a_{j,k}a_{k,j}$, for all distinct j and k such that there is an edge from j to k in Γ.*

(c) *$a_{j,k} = 0$, whenever there is no path from j* to *k* in Γ.

(d) *$a_{j,k} = \frac{a_{j,i}a_{i,k}}{a_{i,i}}$, for all distinct i, j, k, such that there is a path from j to k in Γ, but there is no path from j to k in $\Gamma_i(C^{-1})$.*

The above theorem clearly characterizes, for example, all nonnegative matrices which are inverses of tridiagonal M-matrices. It is quite apparent from the theorem that we have the following theorem.

Theorem 3.3. *Let Γ be a unipathic digraph and let $A \in \mathbb{R}^{n,n}$ an inverse of an M-matrix such that $\Gamma(A^{-1}) = \Gamma$. Then for any function $f : \mathbb{R}_+ \to \mathbb{R}_+$ which is nonnegative, strictly increasing and multiplicative, the $n \times n$ matrix $F = (f(a_{i,j}))$ is a nonnegative matrix whose inverse is an M-matrix with $\Gamma(F^{-1}) = \Gamma$. In particular, $A \circ A$ is an inverse of an M-matrix and $\Gamma((A \circ A)^{-1}) = \Gamma$.*

Proof. Since A is an inverse of an M-matrix for which $\Gamma(A^{-1})$ is the unipathic graph Γ, it is clear, by the equivalence of (i) and (ii) in Theorem 3.2 and by the properties of f that the elements of F satisfy, in turn, all the properties (a)–(d) of (ii) with respect to Γ in Theorem 3.2(ii). The conclusion is now immediate, again by the equivalence in Theorem 3.2. □

We remark the following. We know from the examples in Eqs. (1.9) and (1.10) that in the situation when both A and $B = A \circ A$ are inverses of M-matrices, $\Gamma(A^{-1})$ and $\Gamma(B^{-1})$ *do not necessarily coincide*. However, from the above theorem, we see that *when A is an inverse of an M-matrix and $\Gamma(A^{-1})$ is unipathic, not only is B an inverse of an M-matrix, but $\Gamma(B^{-1}) = \Gamma(A^{-1})$*. Let us give an illustration. The simple cycle is a unipathic graph. Consider then the example of an inverse of an M-matrix already used in [19].

$$A = \begin{bmatrix} 4 & 2 & 2 & 6 & 6 & 8 \\ 2 & 2 & 2 & 6 & 6 & 8 \\ 2 & 1 & 2 & 6 & 6 & 8 \\ 0.3333 & 0.1667 & 0.1667 & 1 & 1 & 1.333 \\ 1 & 0.5 & 0.5 & 1.5 & 3 & 4 \\ 1 & 0.5 & 0.5 & 1.5 & 1.5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & -0.5 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -6 & 0 & 0 \\ 0 & 0 & 0 & 2 & -0.66667 & 0 \\ 0 & 0 & 0 & 0 & 0.66667 & -0.66667 \\ -0.125 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}^{-1}.$$

Then for $B = A \circ A$ we obtain that

$$B = \begin{bmatrix} 16 & 4 & 4 & 36 & 36 & 64 \\ 4 & 4 & 4 & 36 & 36 & 64 \\ 4 & 1 & 4 & 36 & 36 & 64 \\ 0.1111 & 0.02778 & 0.02778 & 1 & 1 & 1.778 \\ 1 & 0.25 & 0.25 & 2.25 & 9 & 16 \\ 1 & 0.25 & 0.25 & 2.25 & 2.25 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 0.083333 & -0.083333 & 0 & 0 & 0 & 0 \\ 0 & 0.33333 & -0.33333 & 0 & 0 & 0 \\ 0 & 0 & 0.33333 & -12 & 0 & 0 \\ 0 & 0 & 0 & 1.3333 & -0.14815 & 0 \\ 0 & 0 & 0 & 0 & 0.14815 & -0.14815 \\ -0.0052083 & 0 & 0 & 0 & 0 & 0.083333 \end{bmatrix}^{-1}.$$

We mention that we did not normalize A in the above example for ease of display. Let $D = \mathrm{diag}(a_{1,1}^{1/2}, \ldots, a_{n,n}^{1/2})$ so that $\tilde{A} = D^{-1}AD^{-1}$ is now a normalized inverse of an M-matrix. Put $\tilde{B} = \tilde{A} \circ \tilde{A}$ so that from the above example, $\tilde{B}$ is also an inverse of an M-matrix with $\tilde{B} \leqslant \tilde{A}$. A computation shows that

$$(\tilde{B})_{3,4} = -24 < -8.4853 = (\tilde{A})_{3,4}.$$

In the examples given in Eqs. (1.9) and (1.10) we showed that when both A and $B = A \circ A$, $B \leqslant A$, are inverses of M-matrices, then not necessarily $A^{-1} \leqslant B^{-1}$ via comparing diagonal entries. Here we see that not necessarily $A^{-1} \leqslant B^{-1}$ can also be demonstrated via comparisons of off-diagonal entries.

Actually we are able to add to the result of Theorem 3.3.

Theorem 3.4. *Let Γ be a unipathic digraph and let $\mathscr{A}(\Gamma)$ be the class of all matrices in $A \in \mathbb{R}^{n,n}$ whose inverse is an M-matrix with $\Gamma(A^{-1}) = \Gamma$. Then $\mathscr{A}(\Gamma)$ is closed under Hadamard multiplication.*

Proof. Let A and B be in $\mathscr{A}(\Gamma)$ so that their respective entries satisfy conditions (ii)(a)–(d) in Theorem 3.3. It is follows by easy checking that the entries of their Hadamard product $A \circ B$ also satisfy conditions (ii)(a)–(d) of that theorem. Hence the conclusion. □

As an illustration for the above theorem consider the following example in which Γ is a unipathic graph with the star graph being its underlying undirected graph. Let

$$A = \begin{bmatrix} 1 & 0.2608 & 0.0109 & 0.4384 \\ 0.2541 & 1 & 0.0028 & 0.1114 \\ 0.3632 & 0.0947 & 1 & 0.1592 \\ 0.4227 & 0.1102 & 0.0046 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.3024 & -0.2793 & -0.0110 & -0.5381 \\ -0.2721 & 1.0710 & 0 & 0 \\ -0.3646 & 0 & 1.0040 & 0 \\ -0.5188 & 0 & 0 & 1.2274 \end{bmatrix}^{-1},$$

$$B = \begin{bmatrix} 1 & 0.4866 & 0.3736 & 0.0959 \\ 0.2142 & 1 & 0.0800 & 0.0205 \\ 0.4734 & 0.2303 & 1 & 0.0454 \\ 0.4990 & 0.2428 & 0.1864 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.3814 & -0.5432 & -0.4538 & -0.1008 \\ -0.2391 & 1.1163 & 0 & 0 \\ -0.5751 & 0 & 1.2146 & 0 \\ -0.5241 & 0 & 0 & 1.0503 \end{bmatrix}^{-1}.$$

Then

$$A \circ B = \begin{bmatrix} 1 & 0.1269 & 0.0041 & 0.0421 \\ 0.0544 & 1 & 0.0002 & 0.0023 \\ 0.1719 & 0.0218 & 1 & 0.0072 \\ 0.2109 & 0.0268 & 0.0009 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.0166 & -0.1278 & -0.0041 & -0.0424 \\ -0.0548 & 1.0070 & 0 & 0 \\ -0.1720 & 0 & 1.0007 & 0 \\ -0.2128 & 0 & 0 & 1.0089 \end{bmatrix}^{-1}.$$

Theorem 3.4 leads to the concluding observations of this section. First, recall that a nonnegative matrix $\in \mathbb{R}^{n,n}$ is called *totally nonnegative* if all its minors are nonnegative. In ([10], Theorem 1) Lewin proves the following equivalence.

Theorem 3.5 (see [10], Theorem 1). *Let $A \in \mathbb{R}^{n,n}$. Consider the following three conditions*:

(i) *A is nonsingular and totally nonnegative.*

(ii) *A is nonsingular and A^{-1} is an M-matrix.*

(iii) *A is nonsingular and A^{-1} is tridiagonal.*

Then any two of the three conditions implies the third.

Suppose now that $A \in \mathbb{R}^{n,n}$ and $B \in \mathbb{R}^{n,n}$ are inverses of a tridiagonal M-matrix with $\Gamma(A^{-1}) = \Gamma(B^{-1}) = \Gamma$. Then Γ is unipathic. Moreover, by Theorem 3.4, $C = A \circ B$ is an inverse of an M-matrix with $\Gamma(C^{-1}) = \Gamma$ and so C^{-1} is a tridiagonal M-matrix. It thus follows by Lewin's theorem quoted above that C too must be totally nonnegative. This then adds to a list of classes of totally nonnegative matrices which are closed under Hadamard multiplication which was drawn up, and partially obtained by, Garloff and Wagner in [7] (see also [23]). We comment that Markham [15] defines a matrix $A = (a_{i,j}) \in \mathbb{R}^{n,n}$ to be a matrix of *type D* if

$$a_{i,j} = \begin{cases} a_i, & i \leqslant j, \\ a_j, & i > j, \end{cases} \qquad \text{where } a_n > a_{n-1} > \cdots > a_1.$$

He has shown that if $a_1 > 0$, then A is a totally nonnegative matrix which is an inverse of a tridiagonal M-matrix. Clearly if A ia a matrix of type D, then so is $B = A \circ A$, and so B is both a totally nonnegative matrix and an inverse of a tridiagonal M-matrix. In fact, that B is totally nonnegative was noticed by Markham in his earlier paper [14]. Thus our observations in Theorem 3.4 and in the foregoing remarks extend Markham's result to the entire class of totally nonnegative matrices whose inverses are M-matrices.

4. The Willoughby inverse

In [24] Willoughby determines sufficient conditions for a positive matrix to be an inverse of an M-matrix. Part of one of his results states as follows.

Theorem 4.1 (see [24], Theorem 2). *Assume that* $0 < y < x < 1$ *and that* $A = (a_{i,j})$ *is an* $n \times n$ *matrix such that* $y \leqslant a_{i,j} \leqslant x$. *Let the interpolation parameter* s *be given by*

$$x^2 = sy + (1 - s)y^2. \tag{4.1}$$

If $n = 2$, *or if* $n > 3$ *and*

$$\frac{1}{n-2} \geqslant s, \tag{4.2}$$

then A^{-1} *is a strictly diagonally dominant (both by row and column) M-matrix.*

For A as in Theorem 4.1 we can now prove the following theorem.

Theorem 4.2. *Let* A *be as in Theorem* 4.1. *Then* $A \circ A$ *is an inverse of a strictly diagonally dominant (both by row and column) M-matrix.*

Proof. Let $B = (b_{i,j}) = A \circ A$. Then $b_{i,i} = 1$, $i = 1, \ldots, n$, and, for $i, j = 1, \ldots, n$, with $i \neq j$, $y^2 \leqslant b_{i,j}^2 \leqslant x^2$. Define the parameter t via

$$x^4 = ty^2 + (1 - t)y^4$$

so that, using Eq. (4.2),

$$t = \frac{x^4 - y^4}{y^2 - y^4} = s\frac{x^2 + y^2}{y + y^2}. \tag{4.3}$$

As $y \leqslant 1$, $y^2 \leqslant y$, and so, by Eqs. (4.1) and (4.2), $x^2 \leqslant y$. But then, $(x^2 + y^2)/(y + y^2) \leqslant 1$ so that $t \leqslant s \leqslant 1/(n-2)$. Putting $\eta = y^2$ and $\xi = x^2$, we see that

$$\xi^2 = t\eta + (1 - t)\eta^2,$$

$$\eta \leqslant b_{i,j} \leqslant \xi, i \neq j, i, j = 1, \ldots, n.$$

The result now follows by Theorem 4.1. $\square$

5. α-Matrices

In Lewin and Neumann [12], the authors characterize the $(0,1)$-matrices whose inverse is an M-matrix. The characterization turns out to be entirely combinatorial. Clearly the Hadamard product of any such matrix with itself is again an inverse of an M-matrix. As mentioned in the introduction, in that paper it is also proved that a *necessary* condition for a nonnegative matrix $A \in \mathbb{R}^{n,n}$ to be an inverse of an M-matrix is that $\Gamma(A) = \overline{\Gamma(A)}$. In [11], Lewin extends some of the results in [12] as follows. A matrix $A \in \mathbb{R}^{n,n}$ is called an *α-matrix* if it is of the form $A = I + \alpha K$, $\alpha \geqslant 0$, where K is a $(0,1)$-matrix with a zero main diagonal. Lewin proves the following theorem.

Theorem 5.1 (see [11], Theorem 6). *Let $A \in \mathbb{R}^{n,n}$ be an α-matrix such that $\Gamma(A)$ is transitive. If $0 \leqslant \alpha \leqslant 1/(n-1)$ when $n > 2$, then A^{-1} is an M-matrix. Moreover, A^{-1} is diagonally dominant in both rows and columns.*

For α-matrices we have the following theorem.

Theorem 5.2 (see [11], Theorem 6). *Let $A \in \mathbb{R}^{n,n}$ be an α-matrix such that $\Gamma(A)$ is transitive. If $0 \leqslant \alpha \leqslant 1/(n-1)$ for $n > 2$. Then $B = A \circ A$ is an inverse of an M-matrix.*

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